International Baccalaureate
Mathematics: Analysis and Approaches SL
Summer 2024

New Classroom Code: kcxdbxo

Please complete each of the following problems before the first day of classes for non-freshman. Each problem should be completed clearly. Make sure that you not only can solve the problems, but understand exactly what has been done. If you are unsure about any of the material, you should refer to your textbook or other resources in order to master the topics.

1. A line has the equations $-7x - 12y + 168 = 0$
    a. Write down the equation of the line in the form $y = mx + c$.
    b. Given that the line intersects the x-axis at point A and the y-axis at point B, find the coordinates of points A and B.
    c. Calculate the area of triangle OAB.

2.
    a. Using your GDC, sketch the curve of $y = -2.9x^2 + 4.1x + 5.9$ for $-1 \leq x \leq 2$.
    b. Write down the coordinates of the points where the curve intersects the x or y axis.
    c. Write down the range of y

3.
    a. Show that the solutions to the equation $x^2 - 6x - 43 = 0$ may be written in the form $x = p \pm q\sqrt{13}$, where p and q are integers.
    b. Hence, or otherwise, solve the inequality $x^2 - 6x - 43 \leq 0$

4. A function f is defined by $f(x) = 3(x-1)^2 - 18,\ x \in \mathbb{R}$
    a. Write $f(x)$ in the form $ax^2 + bx + c$, where a, b, and c are constants
    b. Find the coordinates of the vertex of the graph of f
    c. Find the equation of the axis of symmetry of the graph of f
    d. State the range of f
    e. The graph of f is translated through the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ to form a curve represented by a new function $g(x)$. Find $g(x)$ in the form $px^2 + qx + r$, where p, q, and r are constants

5. The quadratic curve $y = x^2 + bx + c$ intersects the x-axis at (10,0) and has an equation of symmetry $x = 5/2$
   a. Find the values of b and c
   b. Hence, or otherwise, find the other two coordinates where the curve intersects the coordinate axis.

6. Consider the function $f(x) = 2x^2 - 4x - 8,\ x \epsilon \mathbb{R}$
   a. Show that the function can be expressed in the form $f(x) = a(x-h)^2 + k$, where a,h, and k, are constants
   b. The function $f(x)$ may be obtained through a sequence of transformations of $g(x) = x^2$, describe each transformation in turn

7. Consider the equation $f(x) = 2kx^2 + 6x + k,\ x \epsilon \mathbb{R}$
   a. In the case that the equation $f(x) = 0$ has two equal real roots, find the possible value of k
   b. In the case that the equation of the line of symmetry of the curve $y = f(x)$ is $x + 1 = 0$, find the value of k
   c. Solve the equation $f(x) = 0$, when k = 2

8. A curve $y = f(x)$ passes through the points with coordinates $A(-12,10)$, $B(0,16)$, $C(2,9)$ and $D(14,-10)$
   a. Write down the coordinates of each point after the curve has been transformed by $f(x) \rightarrow f(2x)$
   b. Write down the coordinates of each point after the curve has been transformed by $f(x) \rightarrow f(-x)+3$

9.
   a. Find the binomial expansion of $\left(1 - \frac{x}{4}\right)^5$ in ascending powers of x.
   b. Using the first three terms from the above expansion, find the approximation for 0.975^5.

10. The 15th term of an arithmetic series is 143 and the 31st is 183.
   a. Find the first term and the common difference
   b. Find the 100th term of the series

11. Brad deposits $5500 in a savings account which earns 2.75% compound interest per year.

a. Determine how much Brad's investment will be worth after 4 years.
b. Calculate, to the nearest year, how long Brad must wait for the value of the investment to reach $12,000.

12. The coefficient of x^2 in the binomial expansion of $(1+3x)^n$ is 495. Determine the value of n.

13. Find the constant term in the expansion of $\left(x^3 - \frac{2}{x}\right)^8$

14. A convergent geometric series has a sum to infinity of 120. Find the 6th term in the series, given that the common ratio is 0.2.

15. The second term in a geometric series is 180 and the sixth term is $\frac{20}{9}$. Find the sum to infinity of the series.

16. Find the value of $\sum_{n=0}^{n=15} (1.6^n - 12n + 1)$ giving your answer correct to 1 decimal point.

17. A ball is dropped from a vertical height of 20 m. Following each bounce, it rebounds to a vertical height of $\frac{5}{6}$ its previous height. Assuming that the ball continues to bounce indefinitely, show the maximum distance it can travel is 220 m

18. Prove the binomial coefficient identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

19. Find the sum of all integers between 500 and 1400(inclusive) that are not divisible by 7.

20. Consider the function given by $f(x) = \frac{3x-20}{4+2x}$
a. State the largest possible domain for the function.
b. State the largest possible range for the function.
c. Determine the coordinates of any point with a curve $y = f(x)$ intersects the x and y-axes.

21. The function f is defined by $f(x) = \frac{3x+8}{x-1}$, $x \epsilon \mathbb{R}$, $x \neq 1$

a. Write down the equation of the vertical asymptote
b. Write down the equation of the horizontal asymptote
c. Hence, sketch the function, $y = f(x)$

22. The function f is defined by $f(x) = \frac{a+bx}{c+8x}$

a. Given that f has a vertical asymptote at $x = -\frac{3}{4}$, determine the value of c.
b. Given that the curve $y = f(x)$ passes through the points $\left(\frac{1}{2}, \frac{2}{5}\right)$ *and* $\left(4, \frac{-3}{38}\right)$, determine the values of a and b.

23. The number of butterflies kept in a private conservatory may be modeled by the formula $P = \frac{18(1+0.82t)}{3 + 0.034t}$, Where P is the population of butterflies after time t months.

a. Write down the initial butterfly population.
b. Use the formula to estimate the number of butterflies in the conservatory after one year.
c. Calculate the number of months that will have passed when the butterfly population reaches 100.
d. Show that, when governed by this model, the number of butterflies in the conservatory cannot exceed 435.

24. A function is given by $f(x) = 128x - 15, \ -3 < x < 15.$

a. Determine the value of $f\left(\frac{3}{2}\right)$
b. Determine the range of the function f
c. Determine the value of a such that $f(a) = 1162.2$

25. Consider the function $f(x) = \frac{k}{x+1} + 1$, $x > 1$, $x \epsilon \mathbb{R}$, $k \epsilon \mathbb{R}$.

a. Show that $f(x)$ is a self inverse function.
b. State the range of f
c. Sketch the graph of $y = f(x)$

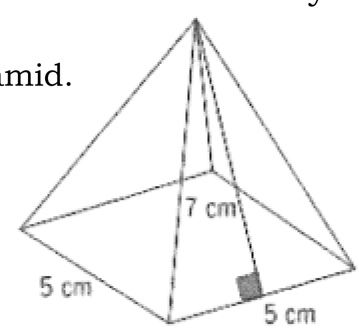
26. Katie organizes a party for her work colleagues. She has a maximum budget of \$1,000. The cost to rent a local hall is \$430 for the evening. She also has to budget for food which will cost approximately \$14.50 per person.

a. Write down a formula connecting the total cost of the party ($C) with the number of people attending the party (p).
b. Explain why $C = f(p)$ is a function.
c. Derive an expression for p in terms of C.
d. Hence, calculate the greatest number of people Katie is able to invite.
e. given that only 16 people attend the party, calculate how much each guest should be charged so that Katie covers her costs.

27. Consider the functions $f(x) = x^2 - 4$, $g(x) = \frac{1}{x+1}$, $h(x) = 2^x$, $x \in \mathbb{R}$
a. Find the range of $f(x)$
b. Find the range of $g(x)$
c. Find the range of $h(x)$
d. Find an expression for $gf(x)$
e. Solve the equation $gf(x) = 9$
f. Solve the inequality $gh(x) > \frac{1}{17}$

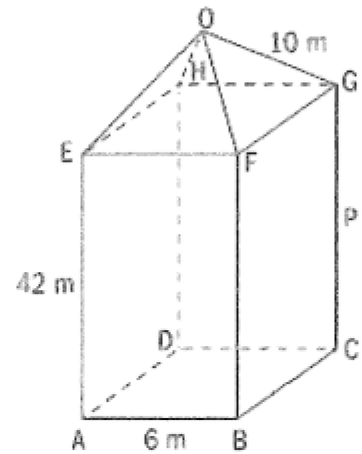
28. Consider the function $p(x) = x^3, \ -2 \leq x \leq 2, \ x \in \mathbb{R}$
a. Find the range of $p(x)$
b. Find an expression for $p^{-1}(x)$.
c. Find all the solutions to the equation $p(x) = p^{-1}(x)$
d. Sketch the graphs of $y = p(x)$ and $y = p^{-1}(x)$ on the same axes.

29. A squared-based pyramid has slant height 7 centimeters. The edges of the base are 5 centimeters long and the apex is located vertically above the center of its base.
a. Find the total surface area of the pyramid.
b. Calculate the volume of the pyramid.



30. A newly built tower is in the shape of a cuboid with a square base. The roof of the tower is in the shape of a square-based right pyramid. The diagram shows the tower and its roof with dimensions OE=OF=OG=OH=10m, AB=BC=CD=AD=6m, and AE=BR=CG=DH=42m.

a. Calculate the the shortest distance from O to EF
b. Hence, find the total surface area of the four triangular sections of the roof.
c. Calculate the height of the tower from the base to O.
d. Determine the size of the angle between OE and EF.



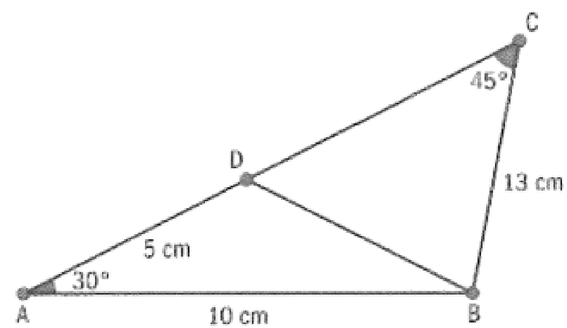
A bird nest is perched at a point P on the edge of CG, of the tower. A person at point B, outside the building, measures the angle of elevation to point P to be 60°.

e. Find the height of the nest from the base of the tower.

31. Consider a trapezium ABCD with AB = CD and BC parallel to AD. Given that AB = 13 cm, BC = 12 cm, and AD = 22 cm,

a. Show that the trapezium has height 12 centimeters.
b. Hence, find the area of the trapezium.
c. Find the size of angle D.
d. Calculate the length of the diagonal AC.

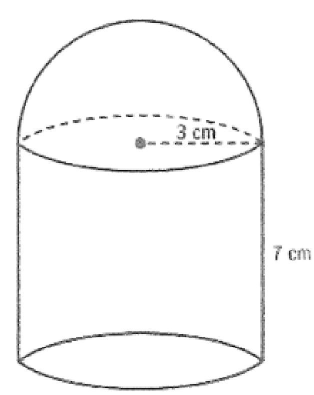
32. The diagram below shows a quadrilateral ABCD such that AB = 10 cm, AD = 5 cm, BC = 13 cm, $B\hat{C}D$ = 45° and $B\hat{A}D$ = 30°.

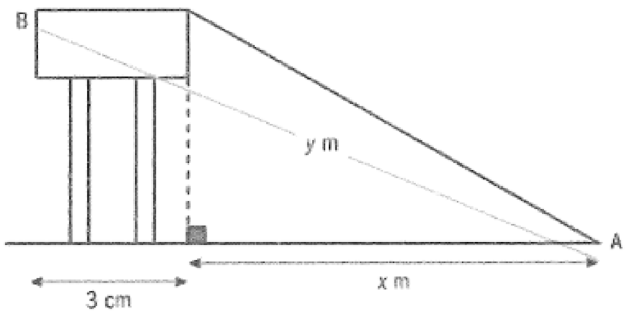


a. Find the exact area of the triangle
b. Show that BD = $5\sqrt{5 - 2\sqrt{3}}$
c. Find the exact value of $sin(\widehat{CDB})$
d. Hence, explain why there are two possible values for the size for the angle $\widehat{CBD}$, stating the relationship between the two possible values.

33. The figure to the right shows a container in the shape of a circular cylinder topped by a hemisphere. The radii of both cylinder and hemisphere is 3 cm and the height of the cylinder is 7 cm. Calculate:

a. The volume of the container.
b. The total surface area of the container.





34. The diagram above shows a water tower with height 30 m and width 3 m. The tower stands on a horizontal platform. From point A on the ground the angle of elevation to the top of the tower is 32°.

   a. Calculate the distance x, giving your answer correct to the nearest meter.
   b. Hence, determine the distance y between A and B.
   c. Find angle of depression A from B.

35. Let $A(t) = 2cos^2t - 3cost + 1,\ 0 \leq t \leq 2\pi$

   a. Factorize *A(t)*
   b. Hence, solve the equation $A(t) = 0$ for $0 \leq t \leq 2\pi$

36. Solve the equation $2cos^2x = sin\,(2x)$ for $0 \leq t \leq 2\pi$, giving your answer in terms of $\boldsymbol{\pi}$.

37. A trigonometric function defined by $f(x) = a\,sin(bx) + c,\ where\ a,\ b,\ and\ c \in \mathbb{R}$. The period of the function f is $\boldsymbol{\pi}$, the maximum of f is 14 and the minimum is 8.

   a. Find the values of the parameters $a,\ b,$ and $c \in \mathbb{R}$.
   b. Hence, sketch a graph of f.

38. Let $S(x) = (sin(2x) + cos(2x))^2$.

   a. Show that $S(x) = sin(4x) + 1$, for all $x \in \mathbb{R}$
   b. Hence, sketch a graph of S for $0 \leq x \leq \pi$
   c. State
      i. The period of S
      ii. The range of S

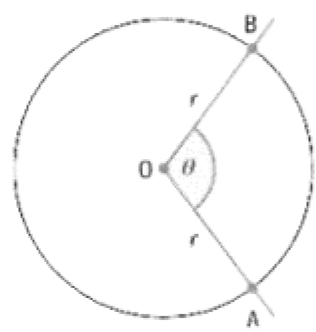
d. Sketch the graph of the function C defined by

$C(x) = cos(2x) - 1, \ for \ 0 \leq x \leq 2\pi$

The graph of C can be obtained from the graph of S under a horizontal stretch of scale factor K followed by a translation by vector $\binom{p}{q}$.

e. Write down
   i. The value of K
   ii. A possible vector $\binom{p}{q}$.

39. The diagram shows a circle with center O and radius r. The points A and B lie on the circumference of the circle. Let $\theta = A\widehat{O}B$



a. If $\theta = \frac{2\pi}{3}$ and $r = 2\ cm$, state in terms of π
   i. The area of sector AOB
   ii. The length of the arc AB
b. Given that the area of the sector AOB is π cm^2 and the length of arc ABC is $\frac{\pi}{3}$ cm, find the exact value of
   i. r
   ii. θ

40. The graph below shows the graph of

$f(x) = atan(b(c - x)) + d, \ \frac{-\pi}{4} < x < \frac{5\pi}{4}, \ x \neq 0, \ x \neq \frac{\pi}{2}, \ x \neq \pi.$

a. State
   i. The equation of the asymptotes to the graph of f
   ii. The period of f
   iii. The range of f
b. Write down the value of the parameters b, and d.
c. Show that a possible value of c is $\frac{\pi}{4}$

Given that the graph contains the point $\left(\frac{\pi}{8}, -2\right)$

d. Show that $a = 3$
e. Write down the solutions to the equation $f(x) = -2$.